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Analytic approximations to Kelvin functions with applications to electromagnetics

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Abstract

We present analytical approximations for the real Kelvin function ber x and the imaginary Kelvin function bei x, using the two-point quasi-fractional approximation procedure. We have applied these approximations to the calculation of the current distribution within a cylindrical conductor. Our approximations are simple and accurate. An infinite number of roots is also obtained with the approximation and precision increases with the value of the root. Our results could find useful applications in problems where analytical approximations of the Kelvin functions are needed.

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1. Introduction

Within the scope of special functions, Kelvin functions appeared early in the literature [1, 2]. They are derived from the Bessel functions [3] of complex argument and many formulae can be found. In addition to their importance in mathematics, they also have a broad spectrum of application in physics; for instance, in the analysis of the current distribution in cylindrical conductors due to wave propagation [4]. A more interesting application is the problem of determining the equivalent impedance of a cylindrical conductor which is given in terms of ber x and bei x. Kelvin functions also arise in the problem of the wave impedance of cylindrically layered conductors for application in nondestructive testing [5]. The temperature distribution in cylindrical conductors due to an alternating current also involves ber x and bei x [6]. Kelvin functions also appear in other fields (i.e. fluid mechanics), but in this paper we will concentrate on the first electromagnetic application mentioned above.

One of the problems we have found in the applications of the Kelvin functions is that their behaviour is rather complicated. Kelvin functions are oscillatory and the amplitude of the oscillations increases rapidly. In spite of the already existing tables, these functions are still a part of integrands where these tables are difficult to use. The programs to compute these functions are also too elaborate.

It seems convenient to have a good approximation to the functions, but the conventional approximation methods do not give good results because of the pathological behaviour of the functions. However, in this paper, it is shown how those difficulties can be surmounted by using the recently published method of two-point quasi-fractional approximations [8, 9]. This has allowed us to find analytical approximations to the Kelvin functions with good accuracy, and furthermore, the same analytical approximation is valid for the whole range of positive values of the variable.

Power series and asymptotic expansions are required in order to obtain these approximations; thus we start analysing those expansions to proceed further. This will be done in section 2 where the form of the approximation will also be derived. Later, the material of this paper will be arranged as follows: the calculation of the parameters for the two-point quasi-fractional approximation of $\operatorname{ber} x$ is carried out in section 3. Section 4 analyses the results of the approximation. The same procedure applied to $\operatorname{ber} x$ in the previous sections is employed in the calculation of $\operatorname{bei} x$. This calculation and the a discussion of the results obtained for $\operatorname{bei} x$ are presented in section 5. In section 6 we show its immediate application to engineering. Finally, section 7 presents a discussion and conclusions of the paper.

2. Two-point quasi-fractional approximation form for the real Kelvin function

It is well known that the real Kelvin function is defined as [2]

$$\operatorname{ber} x = \Re\left(J_0\left(\mathrm{i}\sqrt{\mathrm{i}}\,x\right)\right) \tag{1}$$

where $J_0(z)$ is the zero-order Bessel function. The power series is given by

$$\operatorname{ber} x = 1 - \frac{\left(\frac{1}{4}x^2\right)^2}{(2!)^2} + \frac{\left(\frac{1}{4}x^2\right)^4}{(4!)^2} - \cdots. \tag{2}$$

The asymptotic expansion of $\operatorname{ber} x$ has also been given in several references. Our interest here is mainly in the leading term which can be written as

$$\operatorname{ber} x \sim \frac{\mathrm{e}^{x/\sqrt{2}}\left(\frac{\sqrt{2+\sqrt{2}}}{2}\cos\left(\frac{x}{\sqrt{2}}\right) + \frac{\sqrt{2-\sqrt{2}}}{2}\sin\left(\frac{x}{\sqrt{2}}\right)\right)}{\sqrt{2\pi x}}. \tag{3}$$

Once the potential series (2) and the asymptotic expansion (3) have been obtained, it is possible to determine the form of the quasi-fractional approximations of the function $\operatorname{ber} x$, which has an essential singularity at infinity. That kind of singularity is characteristic of all the hypergeometric confluent functions. The asymptotic expansion (3) picks up this singularity and shows it up through a branch point at infinity together with essential singularities at infinity of the exponential functions. Since the branch points come up in pairs, the leading term of the asymptotic expansion also shows a second branch point at $x = 0$. However, the behaviour of the function $\operatorname{ber} x$ is regular at $x = 0$; therefore, the asymptotic form in (3) is not suitable as an approximation for $\operatorname{ber} x$ in the region near 0. In order to pursue the goal of picking up the right behaviour at infinity, but not to introduce undesired singularities in the zone of interest, a suitable auxiliary function has to be chosen. The auxiliary function should have the right ramification form at infinity and the second branch point should be located outside the zone of interest (i.e. the negative axis).

Only powers of the form $4n$ appear in the potential series of $\operatorname{ber} x$. In order to achieve adequate efficiency in the approximation, we should find such auxiliary functions and fractional approximations whose potential series only have exponents multiples of 4.

Since the second branch point introduced with the asymptotic expansion must be outside the zone of interest, it is possible to choose as an auxiliary function $A_1(x) = 1/\sqrt{1+x}$; however, that auxiliary function would not be efficient since undesired power terms such as $x^2, x^3, \cdots$ would appear. A more suitable auxiliary function would be $\tilde{A}_1(x) = 1/\sqrt[8]{1+x^4}$ or $\tilde{A}_2(x) = 1/\sqrt[8]{1+\tau^4 x^4}$. However, in the last case, the parameter τ could be chosen in a convenient way in order to get better accuracy with the approximation, and as we will discuss later, this can be considered as a free parameter.

The other singularity that must be introduced through the auxiliary functions is of the form $\mathrm{e}^{x/\sqrt{2}} \cos\left(\frac{x}{\sqrt{2}}\right)$ and $\mathrm{e}^{x/\sqrt{2}} \sin\left(\frac{x}{\sqrt{2}}\right)$. Efficiency criteria limit the auxiliary functions by forcing them to have exponents of the form $4n$. The above-stated line of thought leads us to choose as convenient auxiliary functions

$$\cosh\left(\frac{x}{\sqrt{2}}\right) \cos\left(\frac{x}{\sqrt{2}}\right) \tag{4}$$

and

$$\frac{\sinh\left(\frac{x}{\sqrt{2}}\right) \sin\left(\frac{x}{\sqrt{2}}\right)}{x^2}. \tag{5}$$

Once the auxiliary functions are chosen as above, the fractional approximations automatically have only powers that are multiples of 4.

All these previous considerations lead to the following form of the two-point quasi-fractional approximation for the real Kelvin function

$$\tilde{\operatorname{ber}}\, x = \frac{\sum_{k=0}^{n} p_k x^{4k} \cosh\left(\frac{x}{\sqrt{2}}\right) \cos\left(\frac{x}{\sqrt{2}}\right) + \frac{\sqrt{1+\alpha^2 x^4}}{x^2} \sum_{k=0}^{n} P_k x^{4k} \sinh\left(\frac{x}{\sqrt{2}}\right) \sin\left(\frac{x}{\sqrt{2}}\right)}{\left(1 + \sum_{k=1}^{n} q_k x^{4k}\right) \sqrt[8]{1+\tau^4 x^4}}. \tag{6}$$

The auxiliary function $\sqrt{1+\alpha^2 x^4}$ has to be introduced in order to cancel the $1/x^2$ behaviour at infinity. With this function we also define a second free parameter α.

3. Calculation of the two-point quasi-fractional approximation to the real Kelvin function

Here we will consider only the simplest approximation to $\operatorname{ber} x$; thus n will be 1 and the approximation in equation (6) will be reduced to

$$\tilde{\operatorname{ber}}\, x = \frac{\left(p_0 + p_1 x^4\right) \cosh\left(\frac{x}{\sqrt{2}}\right) \cos\left(\frac{x}{\sqrt{2}}\right) + \frac{\sqrt{1+\alpha^2 x^4}}{x^2} \left(P_0 + P_1 x^4\right) \sinh\left(\frac{x}{\sqrt{2}}\right) \sin\left(\frac{x}{\sqrt{2}}\right)}{(1 + q x^4) \sqrt[8]{1+\tau^4 x^4}}. \tag{7}$$

The ideas developed in the latest papers on quasi-fractional approximations are not to determine all the coefficients through the powers series and the asymptotic expansions, but to leave one or two free parameters—α and τ—that can be determined by minimizing the maximum absolute error. It is clear that other methods to measure the discrepancy between the approximant and the exact function can also be used, such as Lebesgue's integral of the quadratic difference, or other more elaborate methods using Lebesgue's integrals of the p-powers of the difference.

However, for us, the most convenient way to measure that discrepancy is the maximum absolute error.

The main reason for using free parameters is to avoid the characteristic defects of classic Padé [7], also common in quasi-fractional approximations. Defects are when an extraneous pole appears near a zero in the numerator. That is, there is a zero in the numerator in x_0 and another zero in the denominator in x_1 and the difference $|x_0 - x_1|$ is very small. Due to that problem, the approximation is in good agreement with the function in the whole zone of interest, except when it is close to x_0 and x_1 where it goes to $+\infty$, $-\infty$ and 0.

In order to determine the five unknowns p_0, P_0, p_1, P_1 and q, three terms of the power series (2) and two terms of the asymptotic expansion (3) will be used. So a linear system of five equations with five unknowns is obtained. The solutions will be given in terms of α and τ.

Looking at the exponential term in (3), the variable $x/\sqrt{2}$ appears more convenient. Thus the power series (2) is written as

$$\mathrm{ber}\left(x\sqrt{2}\right) = 1 - \tfrac{1}{16}x^4 + \tfrac{1}{9216}x^8 - \cdots \tag{8}$$

since we are using only three terms of this series we do not need to go further than x^8.

Now, to compare the power series of $\mathrm{ber}\,x$ and $\tilde{\mathrm{ber}}\,x$, we will first multiply both functions by $(1 + qx^4)$ in order to rationalize the right hand side. Furthermore, the auxiliary functions appearing in (7) and defined in (4) and (5) are replaced by their respective power series. Thus we obtain

$$\begin{aligned}(1+qx^4)\left(1 - \tfrac{1}{16}x^4 + \tfrac{1}{9216}x^8\right) &= \left(1 - \tfrac{1}{8}\tau^4x^4 + \tfrac{9}{128}\tau^8x^8\right)\\ &\quad\times\left(\left(1 - \tfrac{1}{24}x^4 + \tfrac{1}{40320}x^8\right)\left(p_0 + p_1x^4\right) + \left(\tfrac{1}{2} - \tfrac{1}{720}x^4 + \tfrac{1}{3628800}x^8\right)\right.\\ &\quad\left.\times\left(1 + \tfrac{1}{2}\alpha^2x^4 - \tfrac{1}{8}\alpha^4x^8\right)\left(P_0 + P_1x^4\right)\right) + O(x^{12})\end{aligned} \tag{9}$$

where the first term in the first parenthesis on the right hand side is the power expansion of $\sqrt[8]{1+\tau^4x^4}$ and constant factors have been absorbed in the definition of the parameters τ and q.

After equalizing the powers x^0, x^4 and x^8 on both sides of the equation, we obtain

$$p_0 + \tfrac{1}{2}P_0 = 1 \tag{10}$$

$$\left(-\tfrac{1}{24} - \tfrac{1}{8}\tau^4\right)p_0 + \left(-\tfrac{1}{16}\tau^4 - \tfrac{1}{720} + \tfrac{1}{4}\alpha^2\right)P_0 + \tfrac{1}{2}P_1 + p_1 - q = -\tfrac{1}{16} \tag{11}$$

$$\begin{aligned}&\left(\tfrac{1}{192}\tau^4 + \tfrac{1}{40320} + \tfrac{9}{128}\tau^8\right)p_0 + \left(\tfrac{1}{5760}\tau^4 + \tfrac{9}{256}\tau^8 - \tfrac{1}{16}\alpha^4 - \tfrac{1}{32}\tau^4\alpha^2 - \tfrac{1}{1440}\alpha^2 + \tfrac{1}{3628800}\right)P_0\\ &\quad + \left(-\tfrac{1}{24} - \tfrac{1}{8}\tau^4\right)p_1 + \left(-\tfrac{1}{16}\tau^4 - \tfrac{1}{720} + \tfrac{1}{4}\alpha^2\right)P_1 + \tfrac{1}{16}q = \tfrac{1}{9216}.\end{aligned} \tag{12}$$

The leading term of the asymptotic expansion of $\tilde{\mathrm{ber}}\,x$ is

$$\tilde{\mathrm{ber}}\,x \sim \mathrm{e}^{x/\sqrt{2}}\left[\frac{1}{2}\frac{p_1\cos\left(\frac{x}{\sqrt{2}}\right) + P_1\alpha\,\sin\left(\frac{x}{\sqrt{2}}\right)}{q\sqrt{\tau x}}\right]. \tag{13}$$

And after comparing the leading terms of $\mathrm{ber}\,x$, we obtain

$$p_1 = \frac{\sqrt{2+\sqrt{2}}\sqrt{\tau}q}{\sqrt{2\pi}} \tag{14}$$

$$P_1 = \frac{\sqrt{2-\sqrt{2}}\sqrt{\tau}q}{\sqrt{2\pi}\alpha}. \tag{15}$$

Equations (10–12), (14) and (15) determine the values of p_0, P_0, p_1, P_1 and q as functions of α and τ.

When these equations are solved, an expression for each parameter is obtained in terms of α and τ and an approximation $\tilde{\mathrm{ber}}(x, \alpha, \tau)$ is defined for each value of α and τ.

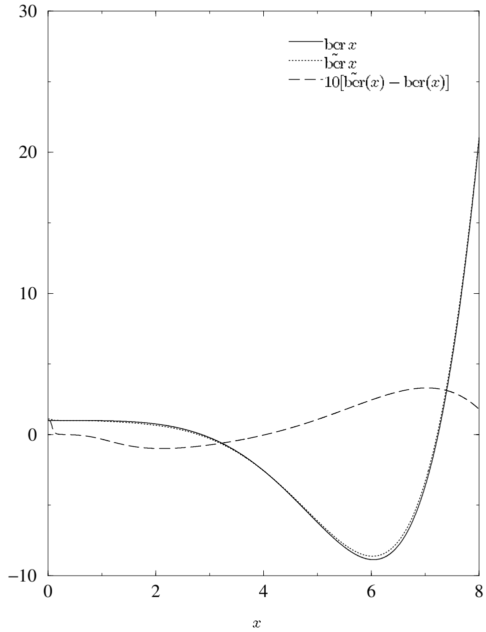

Figure 1. Kelvin real function and its two-point quasi-fractional approximation along with ten times the discrepancy between them.

Given the initial values α_0 and τ_0, we numerically determine the value of the function $|\tilde{\mathrm{ber}}(x, \alpha, \tau) - \mathrm{ber}(x)|$ and select the maximum value of this function which will be the maximum absolute error. Therefore, the maximum absolute error, $\epsilon(\alpha, \tau)$, will be a function of α and τ. Now looking at $\epsilon(\alpha, \tau)$ as a function, we can determine the values α_m and τ_m which minimize this function as a two-variable function. That procedure has been followed and the values α_m and τ_m are found as

$$\alpha_m = 0.98 \qquad \tau_m = 0.8367.$$

In order to avoid the defect in the approximation, we have to consider only positive values of q; thus there is no problem with positive values for the variable x which is the region of interest. This must be taken into account when sweeping through α and τ in the minimizing procedure. Local minima which yield negative values for q must be discarded.

Using these values of α and τ in (10)–(12), (14) and (15), the parameters p_0, P_0, p_1, P_1 and q are determined and the results are

$$\begin{aligned} q &= 27\,627.311\,660 \\ p_0 &= -9750.649\,914 \qquad P_0 = 19\,503.300\,340 \\ p_1 &= 18\,628.544\,300 \qquad P_1 = 7873.669\,071. \end{aligned}$$

4. Accuracy of the two-point quasi-fractional approximation to the real Kelvin function

As has been stated before, the discrepancy in the approximation will be determined by $\Delta \mathrm{ber}\, x = \mathrm{ber}\, x - \tilde{\mathrm{ber}}\, x$.

A plot of the real Kelvin function, together with the approximation and ten times the absolute error is displayed in figure 1.

Table 1 shows the first five roots of the Kelvin function and the introduced approximation.

Table 1. Comparison between the first five roots of $\mathrm{ber}\, x$ and $\tilde{\mathrm{ber}}\, x$.

$\mathrm{ber}\, x$ roots	$\tilde{\mathrm{ber}}\, x$ roots	Relative error (%)
2.848 92	2.786 20	2.2
7.238 83	7.220 30	0.26
11.673 96	11.662 66	0.094
16.113 56	16.105 48	0.052
20.554 63	20.548 34	0.032

Note that the highest relative error in the roots is the first one. Quasi-fractional approximations to any function normally have their worst accuracy for values of order 1. The behaviour of the approximation becomes more accurate as x increases.

Another two measures of the good agreement of the approximation with the real Kelvin function are the relative difference of the position in x of the maxima of the function and the relative error of the amplitude of each maximum. Table 2 shows the position in x of the first five maxima of $\mathrm{ber}\, x$ and $\tilde{\mathrm{ber}}\, x$ as $\mathrm{ber}'\, x$ and $\tilde{\mathrm{ber}}'\, x$, respectively, and the relative difference between them. The value of each of the first five maxima is also shown, appearing as $\mathrm{ber}_m\, x$ and $\tilde{\mathrm{ber}}_m\, x$ for the real Kelvin function and the approximation, respectively. The relative error in those figures is also presented.

Table 2. Comparison between the first five maxima of $\mathrm{ber}\, x$ and $\tilde{\mathrm{ber}}\, x$ and their derivatives.

$\mathrm{ber}'\, x$	$\tilde{\mathrm{ber}}'\, x$	Error (%)	$\mathrm{ber}_m\, x$	$\tilde{\mathrm{ber}}_m\, x$	Error (%)
6.038 71	6.0215	0.28	−8.864 04	−8.614 84	2.9
10.513 64	10.5027	0.10	153.782	151.237	1.7
14.968 44	14.9605	0.053	−2 968.68	−2 933.93	1.2
19.417 58	19.4114	0.032	60 161.2	59 616.8	0.92
23.864 30	23.8592	0.021	$-1.253\,74 \times 10^6$	-1.2445×10^6	0.75

Note that all the relative errors, no matter what is being measured (i.e. the roots, the position of the maxima or the amplitude) decrease as x becomes greater.

5. Calculation of the two-point quasi-fractional approximation for the imaginary Kelvin function

The same line of thought carried out for the real Kelvin function will be followed in this section, since the form of the asymptotic expansions and the power series for both $\mathrm{ber}\, x$ and $\mathrm{bei}\, x$ are very similar.

The imaginary Kelvin function is defined as [2]

$$\mathrm{ber}\, x = \Im(J_0(\mathrm{i}\sqrt{\mathrm{i}}\, x)) \tag{16}$$

where $J_0(z)$ is the zero-order Bessel function. The power series of $\mathrm{bei}\, x$ is given by

$$\mathrm{bei}\, x = \frac{1}{4}x^2 - \frac{\left(\frac{1}{4}x^2\right)^3}{(3!)^2} + \frac{\left(\frac{1}{4}x^2\right)^5}{(5!)^2} - \cdots. \tag{17}$$

The asymptotic expansion of $\mathrm{bei}\, x$ has also been given in several references. Again, our interest here lies mainly in the leading term, which can be written as

$$\mathrm{bei}\, x \sim \frac{\mathrm{e}^{x/\sqrt{2}} \left(\frac{\sqrt{2+\sqrt{2}}}{2} \sin\left(\frac{x}{\sqrt{2}}\right) - \frac{\sqrt{2-\sqrt{2}}}{2} \cos\left(\frac{x}{\sqrt{2}}\right)\right)}{\sqrt{2\pi x}}. \tag{18}$$

It is a characteristic of hypergeometric confluent functions to have an essential singularity at infinity. The asymptotic expansion picks up that singularity and shows it through a branch point at infinity with the exponential functions in the same way as for $\mathrm{ber}\, x$.

The power series of $\mathrm{bei}\, x$ is $x^2, x^6, x^{10}, \ldots$. In order to have an efficient approximation, the auxiliary functions chosen for moving the artificially introduced branch point out of the positive axis and for introducing the singularities that appear in the asymptotic expansion should have a power series of the same form as $\mathrm{bei}\, x$.

Following the same reasoning used in determining the form of $\mathrm{ber}\, x$, a correct form for the quasi-fractional approximation of $\mathrm{bei}\, x$ is

$$\tilde{\mathrm{bei}}\, x = \frac{\frac{x^2}{\sqrt{1+\bar{\alpha}^2 x^4}} \sum_{k=0}^{n} \bar{p}_k x^{4k} \cosh\left(\frac{x}{\sqrt{2}}\right) \cos\left(\frac{x}{\sqrt{2}}\right) + \sum_{k=0}^{n} \bar{P}_k x^{4k} \sinh\left(\frac{x}{\sqrt{2}}\right) \sin\left(\frac{x}{\sqrt{2}}\right)}{\left(1 + \sum_{k=1}^{n} \bar{q}_k x^{4k}\right) \sqrt[8]{1 + \bar{\tau}^4 x^4}} \tag{19}$$

where $\bar{\alpha}$ and $\bar{\tau}$ are the free parameters introduced for the same reason as explained for $\mathrm{ber}\, x$. Here we use only the simplest form of the approximation, thus $n = 1$ will be substituted in the previous equation. Therefore, the two-point quasi-fractional approximation of $\mathrm{bei}\, x$ will give

$$\tilde{\mathrm{bei}}\, x = \frac{\frac{x^2}{\sqrt{1+\bar{\alpha}^2 x^4}} \left(\bar{p}_0 + \bar{p}_1 x^4\right) \cosh\left(\frac{x}{\sqrt{2}}\right) \cos\left(\frac{x}{\sqrt{2}}\right) + \left(\bar{P}_0 + \bar{P}_1 x^4\right) \sinh\left(\frac{x}{\sqrt{2}}\right) \sin\left(\frac{x}{\sqrt{2}}\right)}{(1 + \bar{q} x^4)\sqrt[8]{1 + \bar{\tau}^4 x^4}}. \tag{20}$$

From now on, exactly the same procedure is followed in determining the set of five equations used for finding the values of the parameters $\bar{p}_0$, $\bar{P}_0$, $\bar{p}_1$, $\bar{P}_1$ and $\bar{q}$ in terms of $\bar{\alpha}$ and $\bar{\tau}$ and for minimizing the discrepancy of the approximation.

When dealing with quasi-fractional approximations, it is common to take as an initial ansatz α and τ equal to 1. This choice proved to be a valid starting point for $\tilde{\mathrm{ber}}\, x$, even though smaller differences are obtained when using the minimized $\bar{\alpha}_m$ and $\bar{\tau}_m$ given in the previous section. The $\tilde{\mathrm{bei}}\, x$ function is somewhat more pathological, since the choice $\bar{\alpha} = \bar{\tau} = 1$ yields a negative value of $\bar{q}$, making the initial ansatz useless.

For minimizing the maximum absolute error of the two-point quasi-fractional approximation of $\mathrm{bei}\, x$, a downhill simplex method is used [10]. The values obtained for the parameters are

$$\begin{array}{lll} \bar{\alpha}_m = 3.00 & \bar{\tau}_m = 3.00 & \bar{q} = 19.110\,549\,40 \\ \bar{p}_0 = -7.212\,359\,48 & \bar{P}_0 = 15.424\,718\,96 & \\ \bar{p}_1 = -30.320\,389\,57 & \bar{P}_1 = 24.399\,965\,23. & \end{array}$$

The plots of $\mathrm{bei}\, x$, $\tilde{\mathrm{bei}}\, x$ and ten times the absolute error ($10\Delta\tilde{\mathrm{bei}}\, x = 10(\tilde{\mathrm{bei}}\, x - \mathrm{bei}\, x)$) are shown in figure 2.

The relative error of the first root (table 3) is smaller than that of the first root of $\mathrm{ber}\, x$ because the value of the variable x is greater (i.e. the first root of $\mathrm{bei}\, x$ is not as close to zero as that of $\mathrm{ber}\, x$). The accuracy of all the roots of $\mathrm{ber}\, x$ and $\mathrm{bei}\, x$ is very high in any case.

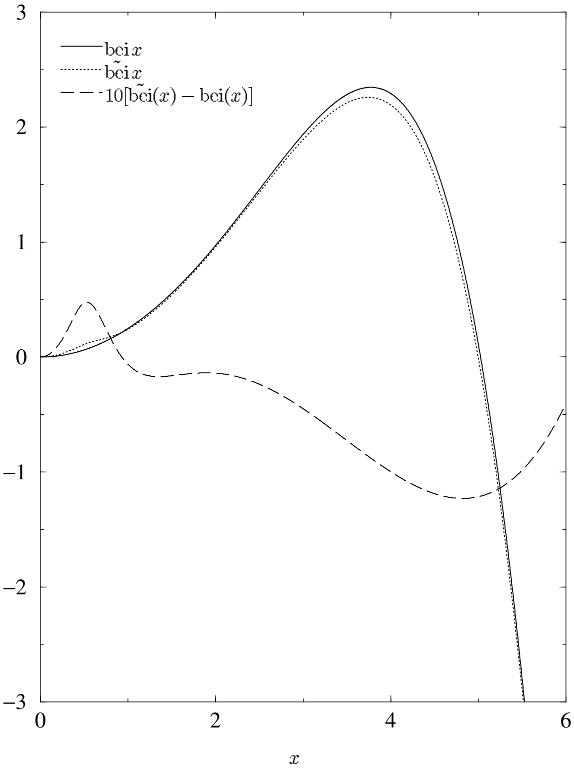

Figure 2. Kelvin imaginary function and its two-point quasi-fractional approximation, along with ten times the discrepancy between them.

Table 3. Comparison between the first five roots of bei x and $\tilde{\text{bei}}\, x$.

bei x roots	$\tilde{\text{bei}}\, x$ roots	Relative error (%)
5.026 22	4.998 73	0.55
9.455 41	9.441 10	0.15
13.893 49	13.884 00	0.069
18.333 98	18.326 89	0.037
22.775 44	22.769 77	0.024

Once again, the relative error related to the position of the first five maxima of the function and the error of their amplitude is calculated in the same way as for ber x, and is presented in table 4. The notation is exactly the same as that followed in the calculation of the error for ber x in section 4 but adapted for the function bei x.

6. Analytical approximation to the current distribution within a cylindrical conductor

The expression for the current distribution within a cylindrical conductor due to a travelling wave is given in many references. The current distribution depends on the Kelvin real and imaginary functions and is given by [4]

$$\left|\frac{J_z(r)}{J_s}\right| = \left(\frac{\text{ber}^2(\sqrt{2}r/\delta) + \text{bei}^2(\sqrt{2}r/\delta)}{\text{ber}^2(\sqrt{2}r_0/\delta) + \text{bei}^2(\sqrt{2}r_0/\delta)}\right)^{1/2} \tag{21}$$

Table 4. Comparison between the first five maxima of $\text{bei}\,x$ and $\widetilde{\text{bei}}\,x$ and their derivatives.

$\text{bei}'\,x$	$\widetilde{\text{bei}}'\,x$	Error (%)	$\text{bei}_m\,x$	$\widetilde{\text{bei}}_m\,x$	Error (%)
3.773 20	3.743 07	0.80	2.346 15	2.259	3.9
8.280 99	8.267 18	0.17	−36.165 4	−35.408 9	2.2
12.742 15	12.732 9	0.073	670.16	660.955	1.4
17.193 43	17.186 5	0.040	−13 305.5	−13 169.6	1.1
21.641 14	21.635 6	0.026	273 888	271 661	0.82

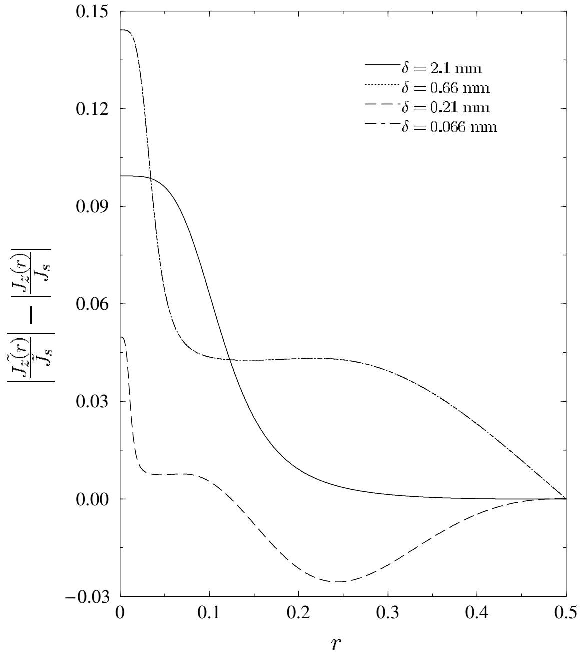

Figure 3. Plots of the discrepancy between the analytical current distribution within the cylindrical conductor and the results obtained with a two-point quasi-fractional approximation for four different values of the skin depth.

where J_s is the current density on the surface of the conductor and r_0 is the radius on the surface. Skin depth is given by δ and it depends on the frequency of the propagating wave (table 5).

By means of the substitution of $\widetilde{\text{ber}}\,x$ and $\widetilde{\text{bei}}\,x$ in (21), it is possible to get an analytical approximation to the current distribution within the cylindrical conductor.

The analytical approximation obtained is used to study the current distribution within a copper wire with $r_0 = 0.5$ mm. Four different values of the frequency are taken. The same figures and plots are found in Marion and Heald [4] for comparison.

The plots obtained using our approximation and those obtained using numerical computation are coincident and no difference can be found at this scale. Therefore, we only show the errors in figure 3.

Table 5. Skin depth at different frequencies.

Case	$\nu = \omega/2\pi$	δ (mm)
1	10^3	2.1
2	10^4	0.66
3	10^5	0.21
4	10^6	0.066

7. Conclusions

In this paper, analytical approximations have been found for the functions ber x and bei x. These have been used to calculate the current distribution within the cylindrical conductor produced by the propagation of one wave of frequency ω.

In spite of the strong fluctuations characteristic of the functions ber x and bei x, the simple approximations found here not only reproduce the function with high accuracy for all positive values of the variable, but also infinite roots or zeros of the function are obtained and the relative error of each root decreases with the magnitude of the root. The largest relative error of the roots for ber x is about 2% and for bei x is 0.5%.

The measure of discrepancy is difficult for these functions because the function fluctuates; and furthermore, the maximum of the amplitude increases and goes to infinity with x. Therefore, if we consider the absolute error this will go to infinity with x.

On the other hand, if we study the relative error, these errors will go to infinity at the roots of the function. The best way to measure the good agreement of our approximation is just through the errors of the zeros and of the amplitude of the oscillations. It is significant that these errors decrease with x despite the large value of the functions. The maximum error of amplitude for ber x is 3%, and 4% for bei x in the value of the function and 0.28% and 0.80% at the site of the maximum for ber x and bei x, respectively.

These errors clearly show that our approximation can be used for most of the applications we know presently; for example, thermo-electrical corrosion in the cylindrical conductors. In general, these approximations are used whenever Kelvin functions appear and a numerical solution is not appropriate. As an example, we apply our method to the simplest case of the current distribution in a cable, obtaining quite accurate results.

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